

MATHEMATICAL FORMULATION OF AN OPTIMAL EXECUTION PROBLEM WITH UNCERTAIN MARKET IMPACT

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ABSTRACT. We study an optimal execution problem with uncertain market impact to derive a more realistic market model. We construct a discrete-time model as a value function for optimal execution. Market impact is formulated as the product of a deterministic part increasing with execution volume and a positive stochastic noise part. Then, we derive a continuous-time model as a limit of a discrete-time value function. We find that the continuous-time value function is characterized by a stochastic control problem with a Lévy process.

1. Introduction

The optimal portfolio management problem is central in mathematical finance theory. There are various studies on this problem, and recently more realistic problems, such as liquidity problems, have attracted considerable attention. In this paper, we focus on market impact (MI), which is the effect of the investment behavior of traders on security prices. MI plays an important role in portfolio theory, and is also significant when we consider the case of an optimal execution problem, where a trader has a certain amount of security holdings (shares of a security held) and attempts to liquidate them before the time horizon. The optimal execution problem with MI has been studied in several papers ([1, 2, 3, 4, 5, 17] and references therein,) and in [11] such a problem is formulated mathematically.

It is often assumed that the MI function is deterministic. This assumption means that we can obtain information about MI in advance. However, in a real market it is difficult to capture the effects of MI without any estimation error. Moreover, it often happens that a high concentration of unexpected orders will result in overfluctuation of the price. The Flash Crash in the United States stock market is a notable example of unusual thinning liquidity: On May 6th, 2010, the Dow Jones Industrial Average plunged by about 9%, only to recover the losses within minutes. Considering the uncertainty in MI, it is thus more realistic and meaningful to construct a mathematical model of random MI. Moazeni et al. [13]

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studied the uncertainty in MI caused by other institutions by compound Poisson processes, and then studied an optimization problem of expected proceeds of execution in a discrete-time setting. They considered the uncertainty in arrival times of large trades from other institutions; however, MI functions of decision makers themselves were given as deterministic linear functions so that the decision makers knew how their own execution affected the market price of the security (the coefficients of MI functions were regarded as “expected price depressions caused by trading assets at a unit rate”).

In this paper, we generalize the framework in [11], particularly considering a random MI function. The model proposed in Section 2 in [11] is derived as a limit of a discrete-time optimal execution problem. Specifically, as in Section A in [11] we first define a discrete-time value function to explicitly describe the situation of each large-volume trade. Then, by taking the limit, we derive the continuous-time version of the value function, which is the main model of [11]. In the present study, we introduce a noise term to a discrete-time MI function to investigate how the effect of uncertainty in the MI function appears in the continuous-time model as a time-scaling limit. We then find that the randomness of MI in the continuous-time model is described as a jump of a Lévy process.

The rest of this paper is organized as follows. In Section 2, we present the mathematical formulation of our model. We set a discrete-time model of an optimal execution problem as our basic model and define the corresponding value function. We also give a convergence theorem of the value functions as our main result. Section 3 contains all the proofs. We briefly conclude this paper in Section 4.

2. The Model and Main Result

In this section, we present the details of the proposed model, which is based on the argument in Section A in [11]. Let (Ω, \mathcal{F}, P) be a complete probability space. $T > 0$ denotes a time horizon, and we assume $T = 1$ for brevity. We assume that the market consists of one risk-free asset (cash) and one risky asset (a security). The price of cash is always 1, which means that a risk-free rate is zero. The price of the security fluctuates according to a certain stochastic flow, and is influenced by sales performed by traders.

First, we consider a discrete-time model with a time interval $1/n$. We consider a single trader who has an endowment of $\Phi_0 > 0$ shares of a security. This trader liquidates the shares Φ_0 over a time interval $[0, 1]$ considering the effects of MI with noise. We assume that the trader sells shares at only times $0, 1/n, \dots, (n-1)/n$ for $n \in \mathbb{N} = \{1, 2, 3, \dots\}$.

For $l = 0, \dots, n$, we denote by S_l^n the price of the security at time l/n , and we also denote $X_l^n = \log S_l^n$. Let $s_0 > 0$ be an initial price (i.e., $S_0^n = s_0$) and $X_0^n = \log s_0$. If the trader sells an amount ψ_l^n at time l/n , the log price changes to $X_l^n - g_l^n(\psi_l^n)$, and by this execution (selling) the trader obtains an amount of cash $\psi_l^n S_l^n \exp(-g_l^n(\psi_l^n))$ as proceeds. Here, the random function

$$g_l^n(\psi, \omega) = c_l^n(\omega)g_n(\psi), \quad \psi \in [0, \Phi_0], \quad \omega \in \Omega$$

denotes MI with noise, which is given by the product of a positive random variable c_l^n and a deterministic function $g_n : [0, \Phi_0] \rightarrow [0, \infty)$. The function g_n is assumed to be non-decreasing, continuously differentiable, and satisfying $g_n(0) = 0$. Moreover, we assume that $(c_l^n)_l$ is independent and identically distributed (i.i.d.), and therefore noise in MI is time-homogeneous. Note that if c_l^n is a constant (i.e., $c_l^n \equiv c$ for some $c > 0$), then this setting is the same as in [11].

After trading at time l/n , X_{l+1}^n and S_{l+1}^n are given by

$$X_{l+1}^n = Y\left(\frac{l+1}{n}; \frac{l}{n}, X_l^n - g_l^n(\psi_l^n)\right), \quad S_{l+1}^n = e^{X_{l+1}^n}, \quad (2.1)$$

where $Y(t; r, x)$ is the solution of the following stochastic differential equation (SDE) on the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t^B)_t, P)$:

$$\begin{cases} dY(t; r, x) = \sigma(Y(t; r, x))dB_t + b(Y(t; r, x))dt, t \geq r, \\ Y(r; r, x) = x, \end{cases}$$

where $(B_t)_{0 \leq t \leq T}$ is a standard one-dimensional Brownian motion (which is independent of $(c_l^n)_l$), $(\mathcal{F}_t^B)_t$ is its Brownian filtration, and $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$ are Borel functions. We assume that b and σ are bounded and Lipschitz continuous, that is,

$$|\sigma(x) - \sigma(y)| + |b(x) - b(y)| \leq K|x - y|, \quad |\sigma(x)| + |b(x)| \leq K, \quad x, y \in \mathbb{R} \quad (2.2)$$

for some $K > 0$. Then, for each $r \geq 0$ and $x \in \mathbb{R}$, there exists a unique solution.

At the end of the time interval $[0, 1]$, the trader has an amount of cash W_n^n and an amount of the security φ_n^n , where

$$W_{l+1}^n = W_l^n + \psi_l^n S_l^n e^{-g_l^n(\psi_l^n)}, \quad \varphi_{l+1}^n = \varphi_l^n - \psi_l^n \quad (2.3)$$

for $l = 0, \dots, n-1$ and $W_0^n = 0$, $\varphi_0^n = \Phi_0$. We say that an execution strategy $(\psi_l^n)_{l=0}^{n-1}$ is admissible if $(\psi_l^n)_l \in \mathcal{A}_n^n(\Phi_0)$ holds, where $\mathcal{A}_k^n(\varphi)$ is the set of strategies $(\psi_l^n)_{l=0}^{k-1}$ such that ψ_l^n is $\mathcal{F}_l^n = \sigma\{(B_t)_{t \leq l/n}, c_0^n, \dots, c_{l-1}^n\}$ -measurable, $\psi_l^n \geq 0$ for each $l = 0, \dots, k-1$ and $\sum_{l=0}^{k-1} \psi_l^n \leq \varphi$ almost surely.

Then, the investor's problem is to choose an admissible strategy to maximize the expected utility $\mathbb{E}[u(W_n^n, \varphi_n^n, S_n^n)]$, where $u \in \mathcal{C}$ is the utility function employed by the investor and \mathcal{C} is the set of non-decreasing, non-negative, and continuous functions on $D = \mathbb{R} \times [0, \Phi_0] \times [0, \infty)$ such that

$$u(w, \varphi, s) \leq C_u(1 + |w|^{m_u} + s^{m_u}), \quad (w, \varphi, s) \in D \quad (2.4)$$

for some constants $C_u, m_u > 0$.

For $k = 1, \dots, n$, $(w, \varphi, s) \in D$ and $u \in \mathcal{C}$, we define the discrete-time value function $V_k^n(w, \varphi, s; u)$ by

$$V_k^n(w, \varphi, s; u) = \sup_{(\psi_l^n)_{l=0}^{k-1} \in \mathcal{A}_k^n(\varphi)} \mathbb{E}[u(W_k^n, \varphi_k^n, S_k^n)]$$

subject to (2.1) and (2.3) for $l = 0, \dots, k-1$ and $(W_0^n, \varphi_0^n, S_0^n) = (w, \varphi, s)$ (for $s = 0$, we set $S_l^n \equiv 0$). We denote such a triplet of processes $(W_l^n, \varphi_l^n, S_l^n)_{l=0}^k$ by $\Xi_k^n(w, \varphi, s; (\psi_l^n)_l)$, and denote $V_0^n(w, \varphi, s; u) = u(w, \varphi, s)$. Then, this problem is equivalent to consider $V_n^n(0, \Phi_0, s_0; u)$. We consider the limit of the value function $V_k^n(w, \varphi, s; u)$ as $n \rightarrow \infty$.

We introduce the following condition for $g_n(\psi)$, which is also assumed in [11].

- [A] $\lim_{n \rightarrow \infty} \sup_{\psi \in [0, \Phi_0]} \left| \frac{d}{d\psi} g_n(\psi) - h(n\psi) \right| = 0$, where $h : [0, \infty) \rightarrow [0, \infty)$ is a non-decreasing continuous function.

Note that in [11], the function $g(\zeta)$ defined by

$$g(\zeta) = \int_0^\zeta h(\zeta') d\zeta' \quad (2.5)$$

represents a MI function in the continuous-time model. In our case, $g(\zeta)$ also corresponds to the strength of MI, but we still must describe the noise in MI.

The following are the conditions for $(c_l^n)_l$:

- [B1] As a definition, $\gamma_n = \text{essinf}_\omega c_l^n(\omega)$. For any $n \in \mathbb{N}$, it holds that $\gamma_n > 0$. In addition,

$$\frac{h(x/\gamma_n)}{n} \rightarrow 0, \quad n \rightarrow \infty \quad (2.6)$$

holds for $x \geq 0$.

- [B2] Let μ_n be the distribution of $(c_0^n + \dots + c_{n-1}^n)/n$. Then, μ_n has a weak limit μ as $n \rightarrow \infty$.

- [B3] There is a sequence of infinitely divisible distributions $(p_n)_n$ on \mathbb{R} such that $\mu_n = \mu * p_n$, and either

[B3-a] $\int_{\mathbb{R}} x^2 p_n(dx) = O(1/n^3)$ as $n \rightarrow \infty$

or

[B3-b] There is a sequence $(K_n)_n \subset (0, \infty)$ such that $K_n = O(1/n)$, $p_n((-\infty, -K_n)) = 0$ (or $p_n((K_n, \infty)) = 0$) and $\int_{\mathbb{R}} x p_n(dx) = O(1/n)$ as $n \rightarrow \infty$,

where O (Landau's symbol) denotes the order notation.

Remark 2.1.

- (i) Let us discuss condition [B1]. First, note that γ_n is independent of l because c_l^n , $l = 0, 1, 2, \dots$ are identically distributed. Next, we examine when the convergence (2.6) holds. Since h is non-decreasing, we see that

$$\frac{h(x/\gamma_n)}{n} \leq \frac{h(\infty)}{n}, \quad n \in \mathbb{N},$$

where $h(\infty) = \lim_{\zeta \rightarrow \infty} h(\zeta) \in [0, \infty]$ (which is well-defined by virtue of the monotonicity of h). This inequality tells us that (2.6) is fulfilled whenever $h(\infty) < \infty$. In the case of $h(\infty) = \infty$, we have the following example:

$$h(\zeta) = \alpha \zeta^p, \quad \gamma_n = \frac{1}{n^{1/p-\delta}} \quad (p, \delta > 0, \delta \leq 1/p). \quad (2.7)$$

We can actually confirm (2.6) by observing that

$$\frac{h(x/\gamma_n)}{n} = \frac{\alpha x^p}{n^{p\delta}} \rightarrow 0, \quad n \rightarrow \infty.$$

Note that [B1] always holds when $\inf_n \gamma_n > 0$, regardless of whether $h(\infty) < \infty$.

- (ii) The condition [B2] holds only when $\liminf_n \gamma_n < \infty$. Indeed, under [B2] we easily see that the support of the distribution μ is included in the interval $[\liminf_n \gamma_n, \infty)$. Note that γ_n of (2.7) satisfies $\limsup_n \gamma_n \leq 1$ because of the relation $\delta \leq 1/p$.
- (iii) Since μ is an infinitely divisible distribution, there is some Lévy process (subordinator) $(L_t)_{0 \leq t \leq 1}$, defined on a certain probability space, such that L_1 is distributed according to μ . To derive the continuous-time model, we want to associate $(c_l^n)_l$ with a difference of $(L_t)_t$, that is, to approximate c_l^n from $n(L_{(l+1)/n} - L_{l/n})$. The condition [B3] implies that the difference between these values is small for large n .

As mentioned in the above remark, there is a Lévy process $(L_t)_t$ such that the distribution of L_1 is μ . Without loss of generality, we may assume that $(L_t)_t$ and $(B_t)_t$ are defined on the same filtered space. Since $(c_l^n)_l$ is independent of $(B_t)_t$, we may also assume that $(L_t)_t$ is independent of $(B_t)_t$. Let ν be the Lévy measure of $(L_t)_t$. Since $(L_t)_t$ is a subordinator, ν satisfies $\nu((-\infty, 0)) = 0$ and either

$$\nu([0, \infty)) < \infty \quad (\text{type A}) \quad (2.8)$$

or

$$\nu([0, \infty)) = \infty, \quad \int_{(0,1)} z\nu(dz) < \infty \quad (\text{type B}). \quad (2.9)$$

See [16] for details. Further, we assume the following moment condition for ν :

$$[\text{C}] \quad \|\nu\|_1 + \|\nu\|_2 < \infty, \quad \text{where } \|\nu\|_p = \left(\int_{(0,\infty)} z^p \nu(dz) \right)^{1/p}.$$

Throughout this paper, we assume [A], [B1]–[B3], and [C].

Now, we define the function that gives the limit of the discrete-time value function. For $t \in [0, 1]$ and $\varphi \in [0, \Phi_0]$ we denote by $\mathcal{A}_t(\varphi)$ the set of $(\mathcal{F}_r)_{0 \leq r \leq t}$ -adapted and caglad processes (i.e., left-continuous and having a right limit at each point) $\zeta = (\zeta_r)_{0 \leq r \leq t}$ such that $\zeta_r \geq 0$ for each $r \in [0, t]$, $\int_0^t \zeta_r dr \leq \varphi$ almost surely and

$$\|\zeta\|_\infty := \sup_{(r,\omega) \in [0,t] \times \Omega} \zeta_r(\omega) < \infty, \quad (2.10)$$

where $\mathcal{F}_r = \sigma\{B_v, L_v; v \leq r\} \vee \{\text{Null sets}\}$. Here, the supremum in (2.10) is taken over all values in $[0, t] \times \Omega$. Note that we may use the essential supremum in (2.10) in place of the supremum.

For $t \in [0, 1]$, $(w, \varphi, s) \in D$ and $u \in \mathcal{C}$, we define $V_t(w, \varphi, s; u)$ by

$$V_t(w, \varphi, s; u) = \sup_{(\zeta_r)_{r \in \mathcal{A}_t(\varphi)}} \mathbf{E}[u(W_t, \varphi_t, S_t)] \quad (2.11)$$

subject to

$$\begin{aligned} dW_r &= \zeta_r S_r dr, \\ d\varphi_r &= -\zeta_r dr, \\ dX_r &= \sigma(X_r) dB_r + b(X_r) dr - g(\zeta_r) dL_r, \\ S_r &= \exp(X_r) \end{aligned} \quad (2.12)$$

and $(W_0, \varphi_0, S_0) = (w, \varphi, s)$. We denote a triplet of processes $(W_r, \varphi_r, S_r)_{0 \leq r \leq t}$ by $\Xi_t(w, \varphi, s; (\zeta_r)_r)$. Note that $V_0(w, \varphi, s; u) = u(w, \varphi, s)$. We call $V_t(w, \varphi, s; u)$ a continuous-time value function. Also note that $V_t(w, \varphi, s; u) < \infty$ for each $t \in [0, 1]$ and $(w, \varphi, s) \in D$.

Remark 2.2. Condition [C] guarantees that the SDE (2.12) has a unique solution for each given $(\zeta_r)_r \in \mathcal{A}_t(\varphi)$ (from Theorem 1.19 in [14]; note that the finiteness of $\|\nu\|_1$ is required for uniqueness). Moreover, by Lemma 3.5 in Section 3, we can show that

$$0 \leq S_r \leq \exp(Y(r; 0, \log s)), \quad r \in [0, t] \quad \text{a.s.},$$

so that, applying Lemma 3.2, for each $m > 0$,

$$\mathbf{E} \left[\sup_{r \in [0, t]} |W_r|^m \right] + \mathbf{E} \left[\sup_{r \in [0, t]} |S_r|^m \right] \leq C_{m, K, \Phi_0} (|w|^m + s^m) \quad (2.13)$$

for some $C_{m, K, \Phi_0} > 0$, where $K > 0$ is as given in (2.2).

Now we give the convergence theorem for value functions.

Theorem 2.3. *For each $(w, \varphi, s) \in D$, $t \in [0, 1]$ and $u \in \mathcal{C}$ it holds that*

$$\lim_{n \rightarrow \infty} V_{[nt]}^n(w, \varphi, s; u) = V_t(w, \varphi, s; u),$$

where $[nt]$ is the greatest integer smaller than or equal to nt .

According to this theorem, a discrete-time value function converges to $V_t(w, \varphi, s; u)$ by shortening the time intervals of execution. This implies that we can regard $V_t(w, \varphi, s; u)$ as the value function of the continuous-time model of an optimal execution problem with random MI. This result is almost the same as in [11], with the exception that the term of MI is given as an increment $g(\zeta_r)dL_r$. Let

$$L_t = \gamma t + \int_0^t \int_{(0, \infty)} z N(dr, dz)$$

be the Lévy decomposition of $(L_t)_t$, where $\gamma \geq 0$ and $N(\cdot, \cdot)$ is a Poisson random measure (see [15, 16], for instance). Then, $g(\zeta_r)dL_r$ can be divided into two terms as follows:

$$g(\zeta_r)dL_r = \gamma g(\zeta_r)dr + g(\zeta_r) \int_{(0, \infty)} z N(dr, dz).$$

The last term on the right side indicates the effect of noise in MI. This means that noise in MI appears as a jump of a Lévy process. Using the above representation and Itô's formula, we see that when $s > 0$ the process $(S_r)_r$ satisfies

$$dS_r = \hat{\sigma}(S_r)dB_r + \hat{b}(S_r)dr - \left\{ \gamma g(\zeta_r)S_r dr + S_{r-} \int_{(0, \infty)} (1 - e^{-g(\zeta_r)z}) N(dz, dr) \right\},$$

where $\hat{\sigma}(s) = s\sigma(\log s)$ and $\hat{b}(s) = s \left\{ b(\log s) + \frac{1}{2}\sigma(\log s)^2 \right\}$ for $s > 0$ (with $\hat{\sigma}(0) = \hat{b}(0) = 0$).

Remark 2.4. It is well known that MI can be divided into two parts: a permanent part and a temporary (or transient) part (see [2, 6] and others). In our study, we mainly treat the permanent MI and do not model the temporary one for the same reason as in Remark 2 in [11]. However, as in [10], we can introduce a price recovery effect by considering, for instance, an Ornstein–Uhlenbeck (OU)-type process such as

$$dX_r = \beta(F_r - X_r)dr + \sigma(X_r)dB_r - g(\zeta_r)dL_r, \quad (2.14)$$

where $\beta > 0$ denotes the speed of price recovery and $(F_r)_r$ is a log-fundamental value process ([10] studies the case where $F_r = \text{Const.}$ and $dL_r = dr$). Then, we can implicitly consider the transient MI in our model. Properties of optimal strategies under the log-price process (2.14) are studied in [7] for the case where we restrict the admissible strategies to deterministic ones and $F_r = \text{Const.}$ We leave the case of adaptive strategies as an area for future study.

Remark 2.5. g describes the shape of the MI function, and assumption [A] implies that g is convex in the wide sense. In practice it is said that the natural form of MI functions is “S-shaped,” that is, concave for small selling and convex for large selling [12]. In this case, the derivative $h = g'$ of the MI function is no longer monotonous. Derivation of an optimal execution problem with an S-shaped deterministic MI function is studied in [12]. In the case of random MI, we further require that γ is strictly positive for technical reason. For details see [7], in which we study the discrete approximation of the continuous-time value function with random MI functions.

Remark 2.6. Theorem 2.3 has the same assertion as Theorem A.1 in [11], and the outlines of our proofs are based on those of [11]. However demonstrating our theorems requires a significant improvement of the proofs. In particular, it is hard to show the L^2 convergence of controlled processes because of a technical difficulty caused by the jump term of $(L_t)_t$. To overcome this problem, we prepare a useful lemma (Lemma 3.3 in Section 3.1) and we give the proofs by properly using both L^1 and L^2 moments to see the convergences of the processes. This is one of the mathematical contributions of this paper. See Section 3.2 and for details. See also Remark 2.2(i) in [8].

3. Proofs

In this section we prepare several lemmas that we use to prove Theorems 2.3. Our approach for the proof is similar to those adopted by [11].

3.1. Preliminaries.

Lemma 3.1. *Let Γ_k ($k \in \mathbb{N}$) be sets, $u \in \mathcal{C}$, and let $(W^i(k, \gamma), \varphi^i(k, \gamma), S^i(k, \gamma)) \in D$ ($\gamma \in \Gamma_k$, $k \in \mathbb{N}$, $i = 1, 2$) be random variables. Assume that*

$$\lim_{k \rightarrow \infty} \sup_{\gamma \in \Gamma_k} \mathbb{E}[|W^1(k, \gamma) - W^2(k, \gamma)|^{m_1} + |\varphi^1(k, \gamma) - \varphi^2(k, \gamma)|^{m_2} + |S^1(k, \gamma) - S^2(k, \gamma)|^{m_3}] = 0$$

and

$$\sum_{i=1}^2 \sup_{k \in \mathbb{N}} \sup_{\gamma \in \Gamma_k} \mathbb{E}[|W^i(k, \gamma)|^{m_4} + (S^i(k, \gamma))^{m_4}] < \infty$$

for some $m_1, m_2, m_3 > 0$ and $m_4 > m_u$, where m_u is as appeared in (2.4). Then we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \sup_{\gamma \in \Gamma_k} & \left| \mathbb{E}[u(W^1(k, \gamma), \varphi^1(k, \gamma), S^1(k, \gamma))] \right. \\ & \left. - \mathbb{E}[u(W^2(k, \gamma), \varphi^2(k, \gamma), S^2(k, \gamma))] \right| = 0. \end{aligned}$$

The above lemma is a generalization of Lemma B.2 in [11]. One can prove Lemma 3.1 by using the Hölder inequality, the Chebyshev inequality, and uniform continuity of $u(w, \varphi, s)$ on any compact set.

Here, we quote Lemma B.1 in [11], as follows, because we frequently use this lemma in the proofs:

Lemma 3.2. *Let $Z(t; r, s) = \exp(Y(t; r, \log s))$ and $\hat{Z}(s) = \sup_{0 \leq r \leq 1} Z(r; 0, s)$. Then, for each $m > 0$, there is a constant $C_{m, K} > 0$ depending only on K and m such that $E[\hat{Z}(s)^m] \leq C_{m, K} s^m$, where $K > 0$ is a constant appearing in (2.2).*

Lemma 3.3. *Let $(X_r^{k, i})_{r \in [0, 1]}$, $i = 1, 2, k \in \mathbb{N}$, be \mathbb{R} -valued $(\mathcal{F}_r)_r$ -progressive processes satisfying*

$$X_r^{k, i} = x^{k, i} + \int_0^r b(X_v^{k, i}) dv + \int_0^r \sigma(X_v^{k, i}) dB_v + F_r^{k, i}, \quad r \in [0, 1],$$

with $x^{k, i} \in \mathbb{R}$ for $i = 1, 2$ and $k \in \mathbb{N}$, where $(F_r^{k, i})_r$ are $(\mathcal{F}_r)_r$ -adapted processes of bounded variation, and let $\Pi_k \subset [0, 1]$, $k \in \mathbb{N}$, be Borel sets. Moreover, assume that

- (i): $x^{k, 1} - x^{k, 2} \rightarrow 0, \quad k \rightarrow \infty,$
- (ii): $\lim_{k \rightarrow \infty} \left\{ D_1^k + \int_0^1 D_r^k dr \right\} = 0,$ where

$$D_r^k = \mathbb{E} \left[\sup_{v \in \Pi_k(r)} |F_v^{k, 1} - F_v^{k, 2}| \right], \quad \Pi_k(r) = ([0, r] \cap \Pi_k) \cup \{r\}.$$

Then it holds that

$$\mathbb{E} \left[\sup_{v \in \Pi_k} |X_v^{k, 1} - X_v^{k, 2}| \right] \rightarrow 0, \quad k \rightarrow \infty.$$

Proof. Define $(\tilde{X}_r^k)_r$ by

$$\tilde{X}_r^k = x^{k, 2} + \int_0^r b(X_v^{k, 1}) dv + \int_0^r \sigma(X_v^{k, 1}) dB_v + F_r^{k, 2}$$

and let

$$\tilde{D}_r^k = \mathbb{E} \left[\sup_{v \in \Pi_k(r)} |\tilde{X}_v^k - X_v^{k, 1}| \right], \quad \Delta_r^k = \mathbb{E} \left[\sup_{v \in \Pi_k(r)} |\tilde{X}_v^k - X_v^{k, 2}|^2 \right].$$

Note that Δ_r^k is finite because of the boundedness of b and σ . We deduce that

$$\mathbb{E}\left[\sup_{v \in \Pi_k(r)} |X_v^{k,1} - X_v^{k,2}|\right] \leq \tilde{D}_r^k + (\Delta_r^k)^{1/2}, \quad r \in [0, 1]. \quad (3.1)$$

Combining the obvious inequality $\tilde{D}_r^k \leq |x^{k,1} - x^{k,2}| + D_r^k$ with (i) and (ii), we see that

$$\tilde{D}_1^k + \int_0^1 \tilde{D}_r^k dr \longrightarrow 0, \quad k \rightarrow \infty. \quad (3.2)$$

Moreover, applying Doob's maximal inequality and the Schwarz inequality, we have that

$$\Delta_r^k \leq 8 \mathbb{E}\left[\int_0^r \left\{|\sigma(X_v^{k,1}) - \sigma(X_v^{k,2})|^2 + |b(X_v^{k,1}) - b(X_v^{k,2})|^2\right\} dv\right]. \quad (3.3)$$

Then we observe that

$$\begin{aligned} & |\sigma(X_v^{k,1}) - \sigma(X_v^{k,2})|^2 \\ & \leq 4K^2 \{1_{\Omega_k(v)} + |\tilde{X}_v^k - X_v^{k,1}|1_{\Omega_k(v)^c} + |\tilde{X}_v^k - X_v^{k,2}|^2 1_{\Omega_k(v)^c}\} \end{aligned}$$

to arrive at

$$\begin{aligned} & \mathbb{E}\left[\int_0^r |\sigma(X_v^{k,1}) - \sigma(X_v^{k,2})|^2 dv\right] \\ & \leq 4K^2 \left\{ \int_0^r P(\Omega_k(v)) dv + \int_0^r \tilde{D}_v^k dv + \int_0^r \Delta_v^k dv \right\} \\ & \leq 8K^2 \left\{ \int_0^r \tilde{D}_v^k dv + \int_0^r \Delta_v^k dv \right\} \end{aligned} \quad (3.4)$$

by using the Chebyshev inequality, where

$$\Omega_k(r) := \left\{ \sup_{v \in \Pi_k(r)} |\tilde{X}_v^k - X_v^{k,1}| > 1 \right\}.$$

Similarly, we get

$$\mathbb{E}\left[\int_0^r |b(X_v^{k,1}) - b(X_v^{k,2})|^2 dv\right] \leq 8K^2 \left\{ \int_0^r \tilde{D}_v^k dv + \int_0^r \Delta_v^k dv \right\}. \quad (3.5)$$

Combining (3.4) and (3.5) with (3.3), we get

$$\Delta_r^k \leq 128K^2 \left\{ \int_0^1 \tilde{D}_v^k dv + \int_0^r \Delta_v^k dv \right\}.$$

Applying the Gronwall inequality, we deduce that

$$\Delta_r^k \leq C \int_0^1 \tilde{D}_v^k dv, \quad r \in [0, 1] \quad (3.6)$$

for some $C > 0$. Our assertion is now obtained from (3.1), (3.2), and (3.6). \square

We can obtain the following lemma, which we need to prove Theorem 2.3 by a standard argument.

Lemma 3.4. *Let $t \in [0, 1]$, $\varphi \geq 0$, $x \in \mathbb{R}$ and $(\zeta_r)_{0 \leq r \leq t} \in \mathcal{A}_t(\varphi)$. Assume further that $(X_r)_{0 \leq r \leq t}$ is given by (2.12) with $(\zeta_r)_r$ and $X_0 = x$. Then, we have*

$$E \left[\sup_{r \in [r_0, r_1]} \left| X_r - X_{r_0} + \int_{r_0}^r g(\zeta_v) dL_v \right|^{2p} \right] \leq \tilde{C}_{p,K} (r_1 - r_0)^p$$

for $p > 0$ and $0 \leq r_0 \leq r_1 \leq t$, where $K > 0$ is a constant appearing in (2.2) and $\tilde{C}_{p,K} > 0$ depend only on p and K .

Arguments similar to the proof of Proposition 5.2.18 in [9] lead us to the following lemma:

Lemma 3.5. *Let $t \in [0, 1]$, $\varphi \geq 0$, $x \in \mathbb{R}$, $(\zeta_r)_{0 \leq r \leq t}, (\zeta'_r)_{0 \leq r \leq t} \in \mathcal{A}_t(\varphi)$ and suppose $(X_r)_{0 \leq r \leq t}$ (resp., $(X'_r)_{0 \leq r \leq t}$) is given by (2.12) with $(\zeta_r)_r$ (resp., $(\zeta'_r)_r$) and $X_0 = x \leq X'_0$. Suppose $\zeta_r \leq \zeta'_r$ for any $r \in [0, t]$ almost surely. Then $X_r \geq X'_r$ for any $r \in [0, t]$ almost surely.*

Note that the above lemma itself can be proved without finiteness of $\|\nu\|_2$.

3.2. Proof of Theorem 2.3. From [B2] and [B3], we see that there exists a Lévy process $(Z_t^n)_t$ which is independent of $(L_t)_t$ and $(B_t)_t$, and that the distribution of Z_1^n is p_n . Then, the stochastic process $L_t^n = L_t + Z_t^n$ also becomes a Lévy process. Now, define $(\tilde{c}_k^n)_k$ by

$$\tilde{c}_l^n = n(L_{(l+1)/n}^n - L_{l/n}^n).$$

Then $(\tilde{c}_k^n)_k$ are i.i.d. random variables with the same distribution as $(c_l^n)_l$. Therefore, $V_k^n(w, \varphi, s; u)$ coincides with $\tilde{V}_k^n(w, \varphi, s; u)$, where $\tilde{V}_k^n(w, \varphi, s; u)$ is the value function defined as the same way as $V_k^n(w, \varphi, s; u)$, replacing $(c_l^n)_l$ with $(\tilde{c}_l^n)_l$. Thus we can identify $(c_l^n)_l$ and $(\tilde{c}_l^n)_l$ without loss of generality (similarly, \mathcal{F}_l^n is identified as $\tilde{\mathcal{F}}_{l/n}$). Let

$$\begin{aligned} C^* &:= \sup_n \left(n \max_{l=0, \dots, n-1} E[|nL_{(l+1)/n} - nL_{l/n} - \tilde{c}_l^n|] \right) \\ &= \sup_n n^2 E[|Z_{1/n}^n|] < \infty. \end{aligned} \tag{3.7}$$

Here, the finiteness of C^* comes from [B3] and the following relations:

$$\begin{aligned} E[(Z_{1/n}^n)^2] &= \frac{1}{n} \int_{\mathbb{R}} x^2 p_n(dx) - \frac{n-1}{n^2} \left(\int_{\mathbb{R}} x p_n(dx) \right)^2, \\ E[|Z_{1/n}^n|] &= \frac{1}{n} \int_{\mathbb{R}} x p_n(dx) + 2 E[(-Z_{1/n}^n) 1_{[0, \infty)}(-Z_{1/n}^n)] \\ &= -\frac{1}{n} \int_{\mathbb{R}} x p_n(dx) + 2 E[Z_{1/n}^n 1_{[0, \infty)}(Z_{1/n}^n)]. \end{aligned}$$

Note that the function g_n on $[0, \Phi_0]$ can be extended on $[0, \infty)$ by

$$g_n(\psi) = g_n(\Phi_0) + \int_{\Phi_0}^{\psi} h(n\psi') d\psi', \quad \psi \in [\Phi_0, \infty).$$

We can now give a proof of Theorem 2.3. We divide the proof into the following two propositions.

Proposition 3.6. $\limsup_{n \rightarrow \infty} V_{[nt]}^n(w, \varphi, s; u) \leq V_t(w, \varphi, s; u)$.

Proposition 3.7. $\liminf_{n \rightarrow \infty} V_{[nt]}^n(w, \varphi, s; u) \geq V_t(w, \varphi, s; u)$.

Proof of Proposition 3.6. For brevity, we assume that $t = 1$. First of all, analogously to the proof of Proposition B.24 in [11], we can show that there exists an optimal strategy $(\hat{\psi}_l^n)_{l=0}^{n-1} \in \mathcal{A}_n^n(\varphi)$ corresponding to the value function $V_n^n(w, \varphi, s; u)$ such that $0 \leq \hat{\psi}_l^n \leq \min\{\psi_n^*, \Phi_0\}$ for each $l = 0, \dots, n-1$, where

$$\psi_n^* = \sup\{\psi \geq 0 ; \gamma_n \psi h_n(\psi) \leq 1\},$$

$h_n = g'_n$, and γ_n is given in [B1]. Set

$$c_n^* = \begin{cases} 2\varepsilon_n + h_n(\psi_n^*) & (h(\infty) = \infty), \\ \varepsilon_n + h(\infty) & (h(\infty) < \infty), \end{cases}$$

where

$$\varepsilon_n = \sup_{\psi \geq 0} \left| \frac{d}{d\psi} g_n(\psi) - h(n\psi) \right|.$$

Then we can prove that

$$\frac{c_n^*}{n} \rightarrow 0, \quad n \rightarrow \infty. \quad (3.8)$$

Indeed, when $h(\infty) < \infty$, (3.8) is obvious from [A]. When $h(\infty) = \infty$, if (3.8) is not true, we see that for each $M > 0$ there is an increasing sequence $(n_k)_k \subset \mathbb{N}$ such that $n_k \gamma_{n_k} \psi_{n_k}^* \leq M$, $k \in \mathbb{N}$ (for brevity we omit k in the notations below). Then we have that $\psi_n^* \leq M/(n\gamma_n)$ and that

$$h_n(\psi_n^*) \leq \varepsilon_n + h(n\psi_n^*) \leq \varepsilon_n + h\left(\frac{M}{\gamma_n}\right).$$

Since $h(\infty) = \infty$ and $\lim_{n \rightarrow \infty} \varepsilon_n = 0$, it holds that $\gamma_n \psi_n^* h_n(\psi_n^*) = 1$ for a sufficiently large n , thus

$$1 \leq \frac{M}{n} \left\{ \varepsilon_n + h\left(\frac{M}{\gamma_n}\right) \right\}.$$

However, [B1] implies that the right side of the above inequality converges to zero as $n \rightarrow \infty$, which leads to a contradiction. Therefore, we get (3.8) and see that

$$g_n(\hat{\psi}_l^n) = \int_0^{\hat{\psi}_l^n} h_n(\psi') d\psi' \leq c_n^* \hat{\psi}_l^n, \quad l = 0, \dots, n-1. \quad (3.9)$$

Remark 3.8.

- (i) In [11], the left side of (3.9) is bounded from above uniformly in n . However, we cannot show the same inequality in our case because of the noise of MI function g_l^n .
- (ii) When $\inf_n \gamma_n > 0$, we can show the uniform boundedness of the left side of (3.9).

To continue the proof of Proposition 3.6, we construct the continuous-time strategy $(\hat{\zeta}_r)_r \in \mathcal{A}_1(\varphi)$ by $\hat{\zeta}_0 = 0$ and $\hat{\zeta}_r = n\hat{\psi}_{\lceil nr-1 \rceil}^n$ ($r > 0$), where $\lceil x \rceil := \min\{n \in \mathbb{Z}; x \leq n\}$ is the ceiling function. Let $(W_l^n, \varphi_l^n, S_l^n)_{l=0}^{n-1} = \Xi_n^n(w, \varphi, s; (\hat{\psi}_l^n)_l)$ and $(W_r, \varphi_r, S_r)_{0 \leq r \leq 1} = \Xi_1(w, \varphi, s; (\hat{\zeta}_r)_r)$, and let $X_l^n = \log S_l^n$ and $X_r = \log S_r$.

Our first step is to apply Lemma 3.3 with

$$F_r^{n,1} = - \int_0^r g(\hat{\zeta}_v) dL_v, \quad F_r^{n,2} = - \sum_{l=0}^{\lceil nr-1 \rceil} g_l^n(\hat{\psi}_l^n)$$

and $\Pi_n = \{l/n; l = 0, \dots, n\}$ to obtain

$$\mathbb{E} \left[\max_{l=0, \dots, n} |X_{l/n} - X_l^n| \right] = \mathbb{E} \left[\sup_{v \in \Pi_n} |X_v - \tilde{X}_v^n| \right] \longrightarrow 0, \quad n \rightarrow \infty, \quad (3.10)$$

where we denote $\sum_{l=0}^{-1} = 0$ and $(\tilde{X}_r^n)_{r \in [0,1]}$ is given by

$$\tilde{X}_r^n = Y\left(r; \frac{k}{n}, X_k^n - g_k^n(\hat{\psi}_k^n)\right), \quad r \in \left(\frac{k}{n}, \frac{k+1}{n}\right] \quad (3.11)$$

and $\tilde{X}_0^n = \log s$. Note that $(\tilde{X}_r^n)_r$ satisfies $\tilde{X}_{l/n}^n = X_l^n$ for $l = 0, \dots, n$ and

$$\tilde{X}_r^n = \log s + \int_0^r \sigma(\tilde{X}_v^n) dB_v + \int_0^r b(\tilde{X}_v^n) dv + F_r^{n,2}.$$

To apply Lemma 3.3, it suffices to show that

$$\mathbb{E} \left[\sup_{v \in \Pi_n} |F_v^{n,1} - F_v^{n,2}| \right] + \int_0^1 \mathbb{E} \left[\sup_{v \in \Pi_n(r)} |F_v^{n,1} - F_v^{n,2}| \right] dr \longrightarrow 0, \quad n \rightarrow \infty. \quad (3.12)$$

A straightforward calculation gives

$$\begin{aligned} \sup_{v \in \Pi_n(r)} |F_v^{n,1} - F_v^{n,2}| &\leq \sum_{l=0}^{n-1} \left| \frac{1}{n} g(n\hat{\psi}_l^n) - g_n(\hat{\psi}_l^n) \right| n(L_{(l+1)/n} - L_{l/n}) \\ &\quad + \sum_{l=0}^{n-1} g_n(\hat{\psi}_l^n) |nL_{(l+1)/n} - nL_{l/n} - \tilde{c}_l^n| \\ &\quad + \mathbf{1}_{[0,1] \setminus \Pi_n}(r) g(n\hat{\psi}_{\lceil nr \rceil}^n) (L_r - L_{\lceil nr-1 \rceil/n}), \quad r \in [0, 1]. \end{aligned} \quad (3.13)$$

From the independence of $\hat{\psi}_l^n$ and $L_{(l+1)/n} - L_{l/n}$ and

$$\sup_{\psi \in (0, \Phi_0]} \left| \frac{g(n\psi)}{n\psi} - \frac{g_n(\psi)}{\psi} \right| \leq \varepsilon_n \longrightarrow 0, \quad \varepsilon \rightarrow 0,$$

we have that

$$\mathbb{E} \left[\sum_{l=0}^{n-1} \left| \frac{1}{n} g(n\hat{\psi}_l^n) - g_n(\hat{\psi}_l^n) \right| n(L_{(l+1)/n} - L_{l/n}) \right] \leq \tilde{\gamma} \Phi_0 \varepsilon_n \longrightarrow 0, \quad n \rightarrow \infty. \quad (3.14)$$

Also, from (3.7) and the independence of $\hat{\psi}_l^n$ and $(L_{(l+1)/n} - L_{l/n}, \tilde{c}_l^n)$, we see that

$$\mathbb{E} \left[\sum_{l=0}^{n-1} g_n(\hat{\psi}_l^n) |nL_{(l+1)/n} - nL_{l/n} - \tilde{c}_l^n| \right] \leq \frac{c_n^*}{n} C^* \Phi_0 \longrightarrow 0, \quad n \rightarrow \infty. \quad (3.15)$$

On the other hand, from the independence of $\hat{\psi}_{[nr]}^n$ and $L_r - L_{[nr-1]/n}$, we can obtain that

$$\begin{aligned} & \mathbb{E} \left[1_{[0,1] \setminus \Pi_n}(r) g(n\hat{\psi}_{[nr]}^n)(L_r - L_{[nr-1]/n}) \right] \\ & \leq \mathbb{E} \left[(c_n^* \hat{\psi}_{[nr]}^n + \Phi_0 \varepsilon_n) n(L_r - L_{[nr-1]/n}) \right] \\ & = \tilde{\gamma}(c_n^* \mathbb{E}[\hat{\psi}_{[nr]}^n] + \Phi_0 \varepsilon_n)(nr - [nr - 1]), \end{aligned}$$

hence

$$\begin{aligned} & \int_0^1 \mathbb{E} \left[1_{[0,1] \setminus \Pi_n}(r) g(n\hat{\psi}_{[nr]}^n)(L_r - L_{[nr-1]/n}) \right] dr \\ & \leq \tilde{\gamma} \Phi_0 \left\{ \frac{c_n^*}{n} + \varepsilon_n \right\} \longrightarrow 0, \quad n \rightarrow \infty. \end{aligned} \quad (3.16)$$

By combining (3.13)–(3.16) we can prove (3.12), and thus we obtain (3.10).

Using the monotonicity of u , we observe that

$$\begin{aligned} V_n^n(w, \varphi, s; u) - V_1(w, \varphi, s; u) & \leq \mathbb{E}[u(W_n^n, \varphi_n^n, S_n^n)] - \mathbb{E}[u(W_1, \varphi_1, S_1)] \\ & \leq \mathbb{E}[u(W_n^n, \varphi_n^n, S_n^n)] - \mathbb{E}[u(\tilde{W}_n^n, \varphi_n^n, S_n^n)] \\ & \quad + \mathbb{E}[u(\tilde{W}_n^n, \varphi_n^n, S_n^n)] - \mathbb{E}[u(W_1, \varphi_1, S_1)], \end{aligned} \quad (3.17)$$

where

$$\begin{aligned} \tilde{W}_n^n & = w + \sum_{l=0}^{n-1} \hat{\psi}_l^n \exp(X_l^n - n(L_{(l+1)/n} - L_{l/n})g_n(\hat{\psi}_l^n)), \\ \tilde{W}_n^n & = w + \sum_{l=0}^{n-1} n\hat{\psi}_l^n \int_{l/n}^{(l+1)/n} \exp(X_l^n - n(L_r - L_{l/n})g_n(\hat{\psi}_l^n)) dr. \end{aligned}$$

Note that $\tilde{W}_n^n \geq \tilde{W}_n^n$ holds almost surely.

From (3.15), Lemma 3.2, and the inequality

$$|e^x - e^y| \leq (e^x + e^y)|x - y|, \quad x, y \in \mathbb{R}, \quad (3.18)$$

we can obtain that

$$\begin{aligned} \mathbb{E}[|\tilde{W}_n^n - W_n^n|^{1/2}] & \leq \widehat{C} \sqrt{\Phi_0} \mathbb{E} \left[\sum_{l=0}^{n-1} g_n(\hat{\psi}_l^n) |n(L_{(l+1)/n} - L_{l/n}) - \tilde{c}_l^n| \right] \\ & \longrightarrow 0, \quad n \rightarrow \infty, \end{aligned} \quad (3.19)$$

where $\widehat{C} = (2sC_{1,K})^{1/2}$ and $C_{1,K}$ is given in Lemma 3.2. Further, applying Lemma 3.4 and using (3.10) and (3.18), we see that

$$\begin{aligned} & \mathbb{E}[|\widetilde{W}_n^n - W_1|^{1/2}] \\ & \leq \widehat{C}\sqrt{\Phi_0} \mathbb{E} \left[\sup_{l=0,\dots,n-1} \sup_{r \in [l/n, (l+1)/n]} |X_r - X_l^n + n(L_r - L_{l/n})g_n(\hat{\psi}_l^n)| \right]^{1/2} \\ & \leq \widehat{C}\sqrt{\Phi_0} \left\{ \widetilde{C}_{2,K} \frac{1}{n^{1/4}} + \mathbb{E} \left[\max_{l=0,\dots,n} |X_{l/n} - X_l^n| \right] \right\}^{1/2} \longrightarrow 0, \quad n \rightarrow \infty. \end{aligned} \quad (3.20)$$

Moreover, obviously it holds that $\varphi_n^n = \varphi_1$ and

$$\mathbb{E}[|S_1 - S_n^n|^{1/2}] \leq \widehat{C} \mathbb{E}[|X_1 - X_n^n|^{1/2}] \longrightarrow 0, \quad n \rightarrow \infty. \quad (3.21)$$

From (3.19)–(3.21), we can apply Lemma 3.1 to see that

$$\lim_{n \rightarrow \infty} |\mathbb{E}[u(\widetilde{W}_n^n, \varphi_n^n, S_n^n)] - \mathbb{E}[u(W_1, \varphi_1, S_1)]| = 0 \quad (3.22)$$

and

$$\lim_{n \rightarrow \infty} |\mathbb{E}[u(W_n^n, \varphi_n^n, S_n^n)] - \mathbb{E}[u(\widetilde{W}_n^n, \varphi_n^n, S_n^n)]| = 0. \quad (3.23)$$

Our assertion is now proved by (3.17), (3.22), and (3.23). \square

Proof of Proposition 3.7. We also assume $t = 1$. Take any $(\zeta_r)_{0 \leq r \leq 1} \in \mathcal{A}_1(\varphi)$ and define $(\psi_l^n)_{l=0}^{n-1} \in \mathcal{A}_n^n(\varphi)$ by

$$\psi_l^n = \int_{(\frac{l-1}{n}) \vee 0}^{\frac{l}{n}} \zeta_r dr.$$

Furthermore, we define $(\hat{\zeta}_v^n)_v \in \mathcal{A}_1(\varphi)$ by $\hat{\zeta}_v^n = 0$ ($0 \leq v \leq 1/n$), $\zeta_{v-\frac{1}{n}}$ ($v > 1/n$). Let $(W_l^n, \varphi_l^n, S_l^n)_{l=0}^{n-1} = \Xi_n^n(w, \varphi, s; (\psi_l^n)_l)$, $(W_r, \varphi_r, S_r)_{0 \leq r \leq 1} = \Xi_1(w, \varphi, s; (\zeta_r)_r)$, and $(\widetilde{W}_r^n, \hat{\varphi}_r^n, \hat{S}_r^n)_{0 \leq r \leq 1} = \Xi_1(w, \varphi, s; (\hat{\zeta}_r^n)_r)$. We also let $X_l^n = \log S_l^n$, $X_r = \log S_r$ and $\hat{X}_r^n = \log \hat{S}_r^n$. Moreover, define $(\tilde{X}_r^n)_r$ by (3.11) replacing $(\hat{\psi}_l^n)_l$ with $(\psi_l^n)_l$.

Since $(\zeta_r)_r$ is left-continuous and bounded, we can apply Lebesgue's dominated convergence theorem to see that

$$\mathbb{E} \left[\sup_{r \in [0,1]} \left| \int_0^r (g(\zeta_v) - g(\hat{\zeta}_v^n)) dL_v \right| \right] \leq \tilde{\gamma} \int_0^1 \mathbb{E} \left[|g(\zeta_v) - g(\hat{\zeta}_v^n)| \right] dv \longrightarrow 0, \quad n \rightarrow \infty.$$

Therefore, we can apply Lemma 3.3 with

$$F_r^{n,1} = - \int_0^r g(\zeta_v) dv, \quad F_r^{n,2} = - \int_0^r g(\hat{\zeta}_v^n) dv$$

and $\Pi_n = [0, 1]$ to obtain

$$\mathbb{E} \left[\sup_{r \in [0,1]} |X_r - \hat{X}_r^n| \right] \longrightarrow 0, \quad n \rightarrow \infty. \quad (3.24)$$

Using Lemma 3.2, (3.18), (3.24), and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} & \mathbb{E}[|W_1 - \hat{W}_1^n|^{1/2}] \\ & \leq \hat{C}' \left\{ \sqrt{\Phi_0} \left(\mathbb{E} \left[\sup_{v \in [0,1]} |X_v - \hat{X}_v^n| \right] \right)^{1/2} + \left(\mathbb{E} \left[\int_0^1 |\zeta_v - \hat{\zeta}_v^n| dv \right] \right)^{1/2} \right\} \\ & \longrightarrow 0, \quad n \rightarrow \infty, \end{aligned} \quad (3.25)$$

where $\hat{C}' = (3sC_{1,K})^{1/2}$.

Next, let $F_r^{n,3} = -\sum_{l=0}^{\lceil nr-1 \rceil} g_l^n(\psi_l^n)$ and $\Pi_n = \{l/n; l = 0, \dots, n\}$. Then we have

$$\begin{aligned} \sup_{v \in \Pi_n(r)} |F_v^{n,2} - F_v^{n,3}| & \leq \sum_{l=0}^{n-1} g_n(\psi_l^n) |nL_{(l+1)/n} - nL_{l/n} - \tilde{c}_l| \\ & \quad + \sum_{l=0}^{n-1} \left| \frac{1}{n} g(n\psi_l^n) - g_n(\psi_l^n) \right| |n(L_{(l+1)/n} - L_{l/n})| \quad (3.26) \\ & \quad + \int_0^1 |g(\hat{\zeta}_v^n) - g(n\psi_{[nv]}^n)| dL_v \\ & \quad + g(\|\zeta\|_\infty) \mathbf{1}_{[0,1] \setminus \Pi_n}(r) (L_{\lceil nr \rceil/n} - L_r), \quad r \in [0, 1]. \end{aligned}$$

Then we see that

$$\begin{aligned} & \mathbb{E} \left[\int_0^1 |g(\hat{\zeta}_v^n) - g(n\psi_{[nv]}^n)| dL_v \right] \\ & \leq \tilde{\gamma} h(\|\zeta\|_\infty) \int_0^{1-\frac{1}{n}} \mathbb{E}[|H_n(v)|] dv \longrightarrow 0, \quad n \rightarrow \infty, \end{aligned} \quad (3.27)$$

where

$$H_n(v) = n \int_{[nv]/n}^{([nv]+1)/n} \zeta_u du - \zeta_v.$$

By (3.26), (3.27), and an argument similar to the proof of Proposition 3.6, we obtain

$$\mathbb{E} \left[\sup_{v \in \Pi_n} |F_v^{n,2} - F_v^{n,3}| \right] + \int_0^1 \mathbb{E} \left[\sup_{v \in \Pi_n(r)} |F_v^{n,2} - F_v^{n,3}| \right] dr \longrightarrow 0, \quad n \rightarrow \infty.$$

Thus we get

$$\mathbb{E} \left[\max_{l=0, \dots, n} |X_l^n - \hat{X}_{l/n}^n| \right] = \mathbb{E} \left[\sup_{v \in \Pi_n} |\tilde{X}_v^n - \hat{X}_v^n| \right] \longrightarrow 0, \quad n \rightarrow \infty \quad (3.28)$$

by virtue of Lemma 3.3.

Define

$$\ddot{W}_n^n = w + n \sum_{l=0}^{n-2} \psi_{l+1}^n \int_{l/n}^{(l+1)/n} \exp(\hat{X}_{v+\frac{1}{n}}^n) dv.$$

Using (3.18), we get

$$\mathbb{E}[|\hat{W}_1^n - \ddot{W}_n^n|^{1/2}] \leq \hat{C}' \left\{ \int_0^{1-\frac{1}{n}} \mathbb{E}[|H_n(v)|] dv \right\}^{1/2} \longrightarrow 0, \quad n \rightarrow \infty. \quad (3.29)$$

Moreover, using Lemma 3.4 and (3.28), we have

$$\begin{aligned} \mathbb{E}[|\ddot{W}_n^n - W_n^n|^{1/2}] &\leq \hat{C}' \mathbb{E} \left[\sum_{l=0}^{n-1} \psi_l^n n \int_{l/n}^{(l+1)/n} |\hat{X}_v^n - X_l^n + g_l^n(\psi_l^n)| dv \right]^{1/2} \\ &\leq \hat{C}' \left\{ \Phi_0 \left[\Phi_0 \varepsilon_n \left(\frac{C^*}{n} + \tilde{\gamma} \right) + \frac{g(\|\zeta\|_\infty)}{n} + \tilde{\gamma} h(\|\zeta\|_\infty) \int_0^{1-\frac{1}{n}} \mathbb{E}[|H_n(v)|] dv \right] \right. \\ &\quad \left. + \Phi_0 \left[\frac{\tilde{C}_{2,K}}{n^{1/4}} + \mathbb{E} \left[\max_{l=0, \dots, n} |X_l^n - \hat{X}_{l/n}^n| \right] + \frac{\tilde{\gamma} \|\zeta\|_\infty g(\|\zeta\|_\infty)}{n} \right] \right\}^{1/2} \\ &\longrightarrow 0, \quad n \rightarrow \infty. \end{aligned} \quad (3.30)$$

By (3.25), (3.29), and (3.30), we get $\lim_{n \rightarrow \infty} \mathbb{E}[|W_1 - W_n^n|^{1/2}] = 0$. Furthermore, using (3.24) and (3.28) we have

$\lim_{n \rightarrow \infty} \mathbb{E}[|X_1 - X_n^n|] = 0$ and $\lim_{n \rightarrow \infty} \mathbb{E}[|S_1 - S_n^n|^{1/2}] = 0$. Moreover, obviously it holds that $\varphi_n^n = \varphi_1$. Then we can apply Lemma 3.1 to obtain

$$\lim_{n \rightarrow \infty} |\mathbb{E}[u(W_1, \varphi_1, S_1)] - \mathbb{E}[u(W_n^n, \varphi_n^n, S_n^n)]| = 0. \quad (3.31)$$

Our assertion is now proved by (3.31) and the following inequality:

$$\mathbb{E}[u(W_1, \varphi_1, S_1)] \leq |\mathbb{E}[u(W_1, \varphi_1, S_1)] - \mathbb{E}[u(W_n^n, \varphi_n^n, S_n^n)]| + V_n^n(w, \varphi, s; u).$$

□

4. Concluding Remarks

In this paper, we generalized the framework in [11] and studied an optimal execution problem with random MI. We defined the MI function as a product of an i.i.d. positive random variable and a deterministic function in a discrete-time model. Furthermore, we derived the continuous-time model of an optimization problem as a limit of the discrete-time models, and found that the noise in MI in the continuous-time model can be described as a Lévy process.

We will investigate properties of the continuous-time value function in our next paper [8].

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